ON HADAMARD PRODUCTS, INTERACTIONS, COVARIATES, AND COMPUTER ROUTINES FOR LINEAR MODELS

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S. R. Searle and Harold V. Henderson*
Biometrics Unit, Cornell University, Ithaca, New York

Abstract

Description is given of the manner in which interactions and covariates are effectively handled in some linear models computer packages using the operation of Hadamard products of columns of the model or design matrix. Consequences of this are discussed.

1. Introduction

Interactions in linear models analyses are sometimes handled by using the matrix operation of the Hadamard product on columns of the incidence matrix corresponding to main effects. In models that include restrictions on the parameters, e.g., the Σ-restrictions where all effects of a factor add to zero, this procedure can lead to erroneous analyses, or sometimes to erroneous computer error messages that analyses cannot be done. The procedure can also be used for fitting covariates in certain models, but it has to be used with care, and interpretation of results must be made accordingly. This paper describes certain features of using Hadamard products in these ways.

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2. Hadamard Products

The Hadamard product of two matrices $\underset{\sim}{A} = \{a_{ij}\}$ and $\underset{\sim}{B} = \{b_{ij}\}$ is $\underset{\sim}{A} * \underset{\sim}{B} = \{a_{ij}b_{ij}\}$ for $i = 1, \cdots, r$ and $j = 1, \cdots, c$, where $*$ is the symbol representing the Hadamard product operation. $\underset{\sim}{A}$ and $\underset{\sim}{B}$ have to be of the same order, $\underset{\sim}{A} * \underset{\sim}{B}$ has that same order, and its (i,j)'th element is the product of the (i,j)'th elements of $\underset{\sim}{A}$ and $\underset{\sim}{B}$. Examples are

$$\underset{\sim}{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad \underset{\sim}{B} = \begin{bmatrix} -4 & 9 \\ 7 & -8 \end{bmatrix}, \qquad \underset{\sim}{A} * \underset{\sim}{B} = \begin{bmatrix} -4 & 18 \\ 21 & -32 \end{bmatrix}$$

and

$$\underset{\sim}{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \underset{\sim}{b} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \underset{\sim}{a} * \underset{\sim}{b} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} .$$

3. Linear Models with Interactions

Hadamard products (hereafter denoted as H-products) of column vectors occur in model equations of linear statistical models that have interactions.

Consider the familiar 2-way crossed classification over-parameterized model

$$E(y_{ijk}) = \mu + \alpha_i + \beta_j + \gamma_{ij} ,$$

where y_{ijk} is the k'th observation in the i'th row and j'th column, μ is a general mean, α_i is the effect due to the i'th row for $i = 1, 2, \cdots, r$, β_j is that due to the j'th column for $j = 1, 2, \cdots, c$, and γ_{ij} is the corresponding interaction effect; $k = 1, 2, \cdots, n_{ij}$ for cells that contain data and $k = 0$ for empty cells. The symbol E represents expectation over repeated sampling. In writing the model as $E(\underset{\sim}{y}) = \underset{\sim}{Xb}$, there will be n_{ij} identical rows of $\underset{\sim}{X}$ corresponding to the n_{ij} observations in row i and column j.

3.1. All cells filled

An example of 2 rows and 3 columns with data in every cell can be represented schematically as in Figure 1, where a check mark indicates presence of data.

Data in Every Cell

✓	✓	✓
✓	✓	✓

Figure 1.

The rows of the $\underset{\sim}{X}$-matrix for this situation are shown in Table 1, where dots represent zeros.

Table 1. Rows of $\underset{\sim}{X}$-matrix Over-parameterized Model — All Cells Filled

No. of Rows	μ	α_1	α_2	β_1	β_2	β_3	γ_{11}	γ_{12}	γ_{13}	γ_{21}	γ_{22}	γ_{23}
n_{11}	1	1	.	1	.	.	1	.	.	.	.	.
n_{12}	1	1	.	.	1	.	.	1	.	.	.	.
n_{13}	1	1	.	.	.	1	.	.	1	.	.	.
n_{21}	1	.	1	1	.	.	.	.	.	1	.	.
n_{22}	1	.	1	.	1	.	.	.	.	.	1	.
n_{23}	1	.	1	.	.	1	.	.	.	.	.	1

In passing, we can note that the body of Table 1 is the $\underset{\sim}{X}$-matrix for the model $E(\underset{\sim}{y}) = \underset{\sim}{X}\underset{\sim}{b}$ for one observation in (or the mean of) each cell.

The H-product operation is apparent in the columns for the γ's: the column for each γ_{ij} is the H-product of the columns for the corresponding α_i and β_j; e.g., that for γ_{11} is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} * \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ 1 \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} .$$

Now suppose a model with Σ-restrictions is used, and to emphasize that this is a different model we use parameters $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$ rather than α, β and γ:

$$\begin{array}{llll}
\dot{\alpha}_1 + \dot{\alpha}_2 = 0 , & \text{i.e.,} & \dot{\alpha}_2 = -\dot{\alpha}_1 & \\
\dot{\beta}_1 + \dot{\beta}_2 + \dot{\beta}_3 = 0 & & \dot{\beta}_3 = -\dot{\beta}_1 - \dot{\beta}_2 & \\
\dot{\gamma}_{11} + \dot{\gamma}_{12} + \dot{\gamma}_{13} = 0 & & \dot{\gamma}_{11} = \dot{\gamma}_{11} & \\
\dot{\gamma}_{21} + \dot{\gamma}_{22} + \dot{\gamma}_{23} = 0 & & \dot{\gamma}_{12} = \dot{\gamma}_{12} & \\
\dot{\gamma}_{11} + \dot{\gamma}_{21} = 0 & & \dot{\gamma}_{13} = -\dot{\gamma}_{11} - \dot{\gamma}_{12} & \\
\dot{\gamma}_{12} + \dot{\gamma}_{22} = 0 & & \dot{\gamma}_{21} = -\dot{\gamma}_{11}, & \dot{\gamma}_{22} = -\dot{\gamma}_{12} \\
\dot{\gamma}_{13} + \dot{\gamma}_{23} = 0 & & \dot{\gamma}_{23} = \dot{\gamma}_{11} + \dot{\gamma}_{12} . &
\end{array}$$

Then the rows of the $\underset{\sim}{X}$-matrix are as shown in Table 2.

Table 2. Rows of $\underset{\sim}{X}$-matrix Σ-restricted Model — All Cells Filled

No. of Rows	$\dot{\mu}$	$\dot{\alpha}_1$	$\dot{\beta}_1$	$\dot{\beta}_2$	$\dot{\gamma}_{11}$	$\dot{\gamma}_{12}$
n_{11}	1	1	1	.	1	.
n_{12}	1	1	.	1	.	1
n_{13}	1	1	-1	-1	-1	-1
n_{21}	1	-1	1	.	-1	.
n_{22}	1	-1	.	1	.	-1
n_{23}	1	-1	-1	-1	1	1

Again, the column for $\dot{\gamma}_{ij}$ is the H-product of the columns for the corresponding $\dot{\alpha}_i$ and $\dot{\beta}_j$.

In this implementation of the Σ-restrictions the last level of each factor is rewritten as minus the sum of the other levels, e.g., $\dot{\beta}_3 = -(\dot{\beta}_1 + \dot{\beta}_2)$. The columns of the $\underset{\sim}{X}$-matrix for each factor in the Σ-restricted model come from that for the unrestricted model by subtracting the column for the last (the eliminated) level from those for the other levels, e.g., that for $\dot{\beta}_1$ in Table 2 is obtained from Table 1 by subtracting from the column for β_1 that for β_3.

3.2. Empty cells

We now consider situations when some cells have no data, and to do so must distinguish between cases in which at least one row and one column are completely filled (i.e., have data in their every cell) and those where no row and/or column are completely filled.

(a) At least one row and one column completely filled

Suppose the 2,3 cell of the preceding example has no data. The schematic representation is then that of Figure 2, where a dash represents an empty cell.

An Empty Cell

√	√	√
√	√	-

Figure 2.

The $\underset{\sim}{X}$-matrix for the over-parameterized model is indicated in Table 3.

Table 3. Rows of $\underset{\sim}{X}$-matrix Over-parameterized Model — An Empty Cell

No. of Rows	μ	α_1	α_2	β_1	β_2	β_3	γ_{11}	γ_{12}	γ_{13}	γ_{21}	γ_{22}
n_{11}	1	1	.	1	.	.	1	.	.	.	.
n_{12}	1	1	.	.	1	.	.	1	.	.	.
n_{13}	1	1	.	.	.	1	.	.	1	.	.
n_{21}	1	.	1	1	.	.	.	.	.	1	.
n_{22}	1	.	1	.	1	.	.	.	.	.	1

Again — the H-product property for columns corresponding to the interaction terms γ_{ij} is very evident.

Now consider the Σ-restrictions for this case:

$$
\begin{array}{llll}
\dot{\alpha}_1 + \dot{\alpha}_2 = 0\,, & \text{rewritten as} & \dot{\alpha}_2 &= -\dot{\alpha}_1 \\
\dot{\beta}_1 + \dot{\beta}_2 + \dot{\beta}_3 = 0 & & \dot{\beta}_3 &= -\dot{\beta}_1 - \dot{\beta}_2 \\
\dot{\gamma}_{11} + \dot{\gamma}_{12} + \dot{\gamma}_{13} = 0 & & \dot{\gamma}_{11} &= \dot{\gamma}_{11} \\
\dot{\gamma}_{21} + \dot{\gamma}_{22} = 0 & & \dot{\gamma}_{12} &= -\dot{\gamma}_{11} \\
\dot{\gamma}_{11} + \dot{\gamma}_{21} = 0 & & \dot{\gamma}_{21} &= -\dot{\gamma}_{11} \\
\dot{\gamma}_{12} + \dot{\gamma}_{22} = 0 & & \dot{\gamma}_{22} &= \dot{\gamma}_{11} \\
\dot{\gamma}_{13} = 0 & & \dot{\gamma}_{13} &= 0\,,
\end{array}
$$

expressing the last level of each factor in terms of the others. The rows of the $\underset{\sim}{X}$-matrix are shown in Table 4.

Table 4. Rows of $\underset{\sim}{X}$-matrix Σ-restricted Model — An Empty Cell

No. of Rows	$\dot{\mu}$	$\dot{\alpha}_1$	$\dot{\beta}_1$	$\dot{\beta}_2$	$\dot{\gamma}_{11}$
n_{11}	1	1	1	.	1
n_{12}	1	1	.	1	-1
n_{13}	1	1	-1	-1	.
n_{21}	1	-1	1	.	-1
n_{22}	1	-1	.	1	1

Notice that in this, the empty cell case, the column in the $\underset{\sim}{X}$-matrix corresponding to $\dot{\gamma}_{11}$ is not the H-product of the columns corresponding to $\dot{\alpha}_1$ and $\dot{\beta}_1$. Furthermore, even though in Table 4 there is a $\dot{\beta}_2$, there is no $\dot{\gamma}_{12}$, and so the H-product of columns corresponding to $\dot{\alpha}_1$ and $\dot{\beta}_2$ does not enter into consideration.

The important conclusion here is that for data with empty cells, and using Σ-restricted models with interactions, columns of the $\underset{\sim}{X}$-matrix corresponding to interactions cannot always be generated as H-products of the columns for the corresponding levels of the main effects. In accord with this conclusion, suppose the H-product procedure was used blindly on the $\dot{\alpha}_1$, $\dot{\beta}_1$ and $\dot{\beta}_2$ columns of Table 4. The result would be Table 5.

Table 5. Rows of A Wrong $\underset{\sim}{X}$-matrix Σ-restricted Model Using H-products — An Empty Cell

No. of Rows	$\dot{\mu}$	$\dot{\alpha}_1$	$\dot{\beta}_1$	$\dot{\beta}_2$	$\dot{\gamma}^+_{11}$	$\dot{\gamma}^+_{12}$
n_{11}	1	1	1	.	1	.
n_{12}	1	1	.	1	$\boxed{.}$	1
n_{13}	1	1	-1	-1	$\boxed{-1}$	-1
n_{21}	1	-1	1	.	-1	.
n_{22}	1	-1	.	1	$\boxed{.}$	-1

Compared to Table 4 this is wrong on two counts: it has too many columns, and even after deleting the last column the $\dot{\gamma}^+_{11}$ column of Table 5 is not the $\dot{\gamma}_{11}$ column of Table 4. The "boxed in" elements in Table 5 are wrong.

Although not all cells are filled in the data pattern of Figure 2, there is both a row and a column that does have all cells filled. If the levels corresponding to a filled row and column are rewritten as minus the sum of the other levels, so that the corresponding columns in the $\underset{\sim}{X}$-matrix of the unrestricted model are subtracted from other columns for that factor, then the H-product procedure can be made to work for the Σ-restricted model. For example, row 1 and column 2 in Figure 2 both have all cells filled. Therefore we can rewrite $\dot{\alpha}_1$ and $\dot{\beta}_2$ in terms of the other $\dot{\alpha}$'s and $\dot{\beta}$'s respectively, as $\dot{\alpha}_1 = -\dot{\alpha}_2$ and $\dot{\beta}_2 = -(\dot{\beta}_1 + \dot{\beta}_3)$. Then it is easily established that the rows of the $\underset{\sim}{X}$-matrix for this implementation of the Σ-restrictions can be as given in Table 6.

Table 6. Rows of $\underset{\sim}{X}$-matrix for Data of Figure 3 With A Σ-restricted Model

No. of Rows	$\dot{\mu}$	$\dot{\alpha}_2$	$\dot{\beta}_1$	$\dot{\beta}_3$	$\dot{\gamma}_{21}$
n_{11}	1	-1	1	.	-1
n_{12}	1	-1	-1	-1	1
n_{13}	1	-1	.	1	.
n_{21}	1	1	1	.	1
n_{22}	1	1	-1	-1	-1

The column for $\dot{\gamma}_{21}$ is clearly the H-product of the columns for $\dot{\alpha}_2$ and $\dot{\beta}_1$. The full H-product procedure would also yield a $\dot{\gamma}_{23}$ column, which is superfluous. To the extent that the H-product procedure yields correct columns except for there being too many of them, the procedure works in this case; but only when the effects for a filled row and filled column of the data are expressed in terms of other effects as just described.

(b) No row or column completely filled

It seems to us that when every row and column has at least one empty cell the H-product procedure can never be made to work. Consider the following example.

A Data Pattern With No Row or Column Completely Filled

√	-	√	√
√	√	-	-
-	√	√	√

Figure 3.

The rows of the $\underset{\sim}{X}$-matrix for a Σ-restricted model for this case are shown in Table 7.

Table 7. Rows of the $\underset{\sim}{X}$-matrix for Data of Figure 3, With A Σ-restricted Model

No. of Rows	$\dot{\mu}$	$\dot{\alpha}_1$	$\dot{\alpha}_2$	$\dot{\beta}_1$	$\dot{\beta}_2$	$\dot{\beta}_3$	$\dot{\gamma}_{11}$	$\dot{\gamma}_{13}$
n_{11}	1	1	.	1	.	.	1	.
n_{13}	1	1	.	.	.	1	.	1
n_{14}	1	1	.	-1	-1	-1	-1	-1
n_{21}	1	.	1	1	.	.	$\boxed{-1}$	.
n_{22}	1	.	1	.	1	.	$\boxed{1}$	.
n_{32}	1	-1	-1	.	1	.	$\boxed{-1}$	.
n_{33}	1	-1	-1	.	.	1	.	-1
n_{34}	1	-1	-1	-1	-1	-1	1	1

It is clear that the $\dot{\gamma}_{11}$ column is not the H-product of the columns for $\dot{\alpha}_1$ and $\dot{\beta}_1$. If it were, the three "boxed" elements in Table 7 would be zero.

(c) The (SAS) HARVEY computing routine

This routine has a feature which is pertinent to the preceding two sections. First, for data like those envisaged in Figure 2, having at least one row and one column with all cells filled, the HARVEY procedure requires that rows and columns each be resequenced (if necessary) so that the last row and column in the new sequence have all cells filled.

Second, when no row and/or column has all cells filled, the HARVEY procedure in the example of Figure 3 uses zero for the "boxed" elements of Table 7. This is shown on page 46 of Searle and Henderson [1978b], the Annotated Computer Output for this routine, with n_{ij} repetitions of the rows as shown in Table 7. It then yields a misleading error message that the data cannot be analyzed.

4. The 1-way Classification with Covariate

4.1. The 1-way classification model

The usual model for the 1-way classification is

$$E(y_{ij}) = \mu + \alpha_i \tag{1}$$

where y_{ij} is the j'th observation on the i'th class, μ is a general mean, α_i is an effect due to the i'th class, and E represents expectation over repeated sampling; for a classes and n_i observations in the i'th class, we have $j = 1, \cdots, n_i$ and $i = 1, \cdots, a$. The familiar matrix representation of (1) is

$$E(\underset{\sim}{y}) = \underset{\sim}{X}\underset{\sim}{a} \tag{2}$$

where $\underset{\sim}{y}$ is the vector of observations, $\underset{\sim}{a}$ is the vector of μ and the α_i's, and $\underset{\sim}{X}$ is the corresponding incidence matrix of 0's and 1's.

For covariance, using the letter z for covariates in order to maintain the traditional use of $\underset{\sim}{X}$, the usual model corresponding to (1) is

$$E(y_{ij}) = \mu + \alpha_i + bz_{ij} , \tag{3}$$

and the form corresponding to (2) is, as in Searle [1971, Chapter 8],

$$E(\underset{\sim}{y}) = \underset{\sim}{X}\underset{\sim}{a} + \underset{\sim}{Z}\underset{\sim}{b} . \tag{4}$$

The model for having a different "slope" in each class is

$$E(y_{ij}) = \mu + \alpha_i + b_i z_{ij} . \tag{5}$$

Although the analysis for the model (5) is well-known in the form of fitting several regressions, the manner of its execution by certain computer routines, for example, SAS GLM, bears discussion.

First note that (5) can be written in a manner that explicitly includes (3) in the form

$$E(y_{ij}) = \mu + \alpha_i + (b + b_i^*)z_{ij} \; . \tag{6}$$

This will be a non full rank model, not only insofar as μ and the α_i's are concerned but also in regard to the b and b_i^*'s. For example, suppose the z's for five observations are 1, 2 and 3 in the first class and 4 and 5 in the second. Then (4) for the model (3) is

$$E(\underset{\sim}{y}) = \begin{bmatrix} 1 & 1 & . \\ 1 & 1 & . \\ 1 & 1 & . \\ 1 & . & 1 \\ 1 & . & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} b \tag{7}$$

whereas for (6) it is

$$E(\underset{\sim}{y}) = \begin{bmatrix} 1 & 1 & . \\ 1 & 1 & . \\ 1 & 1 & . \\ 1 & . & 1 \\ 1 & . & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & . \\ 2 & 2 & . \\ 3 & 3 & . \\ 4 & . & 4 \\ 5 & . & 5 \end{bmatrix} \begin{bmatrix} b \\ b_1^* \\ b_2^* \end{bmatrix} . \tag{8}$$

The first 5 x 3 matrix is not of full column rank; neither is the second.

4.2. The SAS GLM computing procedure

(a) User specifications

The SAS GLM computer routine can handle the model (5) directly; but to provide a test of the hypothesis that in (5) the b_i's are all equal, the model (6) must be used. For (5) and (6) the user must specify the "interaction" Z * A, where * is now a SAS symbol and not an H-product. Although as a computing specification this looks like a regular interaction it is not, for an unrestricted model, an interaction of factors in the usual sense (and especially with regard to the Type II sums of squares coming from SAS).

The specification Z * A for models (5) and (6) effectively generates columns for the $\underset{\sim}{Z}$-matrix as H-products of columns corresponding to those of Z and A in the model (3). This is clearly evident in (8): the columns of $\underset{\sim}{Z}$ in (8) are H-products of those of $\underset{\sim}{X}$ and $\underset{\sim}{Z}$ in (7). This H-product operation is well understood as a method of generating columns of an incidence matrix corresponding to interactions of factors, e.g., A * B for factors A and B. But when either A, or B, or both, represent covariates the specification A * B, e.g., Z * A, still yields calculations appropriate for models like (5) and (6), but then Z * A is no longer an interaction in the usual sense. Other examples are Z * Z and Z * Z * Z for fitting second and third order polynomials of a covariate.

The columns of an incidence matrix for an unrestricted model generated by A * B when A and B are factors are linearly dependent upon those corresponding to A and to B. (Tables 1 and 3 contain examples.) In contrast, the manner in which Z * A is not an interaction in an unrestricted model is that the columns which Z * A generates are not linearly dependent upon those corresponding to A. (Similarly, columns generated by Z * Z and Z * Z * Z are not linearly dependent upon those corresponding to Z.) For example, in (8), the columns of the second matrix are not linearly dependent upon columns of the first. It is this feature of the "interaction" Z * A that leads to interpretation of SAS output being a little different from that for interactions in the regular sense. Description of some of that output now follows.

(b) Sums of squares

Description of the sums of squares is in terms of the model (6) using the traditional symbol x_{ij} for the covariate:

$$E(y_{ij}) = \mu + \alpha_i + (b + b_i^*)x_{ij} \text{ ,}$$

where the vector $\underset{\sim}{b}^*$ is defined as

$$\underset{\sim}{b}^* = [b_1^* \quad b_2^* \quad \cdots \quad b_a^*]' \text{ .}$$

Notes: (1) This model is the same as

$$E(y_{ij}) = \mu + \alpha_i + b_i x_{ij}$$

with

$$b_i = b + b_i^* \text{ .}$$

(2) By definition, all cells are filled. No group in a 1-way classification is without data; if it is, it is simply not included in the model.

(3) Each line of output in the following description of the sums of squares is numbered to provide easy reference.

(4) When including Σ-restrictions in the model we use overhead dots on the parameters to distinguish that model from the unrestricted model; e.g., $\dot{\alpha}_i$ instead of α_i .

Table 8. Sums of Squares from SAS GLM for Fitting a 1-way Classification Model with Separate Slopes

No.	Source	d.f.	Sum of Squares	Hypothesis tested when sum of squares is used in numerator of an F-statistic
	(A) Fitting the sequence A, X, X * A			
	TYPE I			
1	A	a-1	$R(\alpha\|\mu)$	$H: \alpha_i + (b + b_i^*)\bar{x}_{i.}$ equal for all i [See Searle (1971), p. 358]
2	X	1	$R(b\|\mu,\alpha)$	$H: \Sigma k_i(b + b_i^*) = 0$ for $\Sigma k_i = 1$ [See (55), Searle (1971), p. 358]
3	X * A	a-1	$R(\underset{\sim}{b}^*\|\mu,\alpha,b)$	$H: b_i^*$'s all equal [i.e., b_i's all equal, where $b_i = b + b_i^*$]

Table 8 (continued)

No.	Source	d.f.	Sum of Squares	Hypothesis tested when sum of squares is used in numerator of an F-statistic
	TYPE II			
4	A	a-1	$R(\alpha \mid \mu, b, \underset{\sim}{b}^*)$ [This is adjusted for $\underset{\sim}{b}^*$, even though $\underset{\sim}{b}^*$ is introduced computationally as an "interaction"; but it is not an interaction in the usual sense.]	$H: \alpha_i$'s all equal
5	X	1	$R(b \mid \mu, \alpha)$	See line 2
6	X * A	a-1	$R(\underset{\sim}{b}^* \mid \mu, \alpha, b)$	See line 3
	TYPE III			
		Type III sums of squares are for each factor adjusted for all others using the Σ-restrictions. In the presence of the H-product operator involving the b^*'s, this has the effect of then making the $\dot{b}^*$'s add to zero, so removing the non full rank feature of the $(b + b^*_i)x_{ij}$ part of the model.		
7	A	a-1	$R^*(\dot{\alpha} \mid \dot{\mu}, \dot{b}, \dot{\underset{\sim}{b}}^*)_\Sigma$ [This is the same as line 4 because the hypotheses $H: \dot{\alpha}_i = 0$ and $H: \alpha_i$'s all equal are equivalent.]	$H: \dot{\alpha}_i = 0$ for all i, equivalent to $H: \alpha_i$'s all equal.
8	X	1	$R^*(\dot{b} \mid \dot{\mu}, \dot{\alpha}, \dot{\underset{\sim}{b}}^*)_\Sigma$	$H: \dot{b} = 0$, equivalent to $H: \sum_i (b + b^*_i)/a = 0$
9	X * A	a-1	$R^*(\dot{\underset{\sim}{b}}^* \mid \dot{\mu}, \dot{\alpha}, \dot{b})_\Sigma = R(\underset{\sim}{b}^* \mid \mu, \alpha, b)$	See line 3
	TYPE IV			
		The same as TYPE III, because all cells are filled.		
	(B) Fitting the sequence X, A, X * A			
	TYPE I			
10	X	1	$R(b \mid \mu)$	$H: \Sigma k_i(b + b^*_i) + \Sigma c_i \alpha_i = 0$ with $\Sigma k_i = 1$ and $\Sigma c_i = 0$
11	A	a-1	$R(\alpha \mid \mu, b)$	$H: \Sigma \ell_i b^*_i + \Sigma d_i \alpha_i = 0$ for some ℓ_i and $\Sigma d_i = 0$
12	X * A	a-1	$R(\underset{\sim}{b}^* \mid \mu, \alpha, b)$	Same as line 3

Only Type I sums of squares are affected by the sequence in which factors (and/or covariates) are presented as input. This is why Part B of Table 8 shows only Type I sums of squares.

Examples of the sums of squares of Part A of Table 8 are shown on page 60 of Searle and Henderson [1978a], the Annotated Computer Output for SAS GLM. (Parts of the output on page 60 of the December 1978 and May 1979 versions are poorly and some wrongly labeled – updates are now available.)

5. The 2-way Crossed Classification with Covariate

In a manner similar to that for the 1-way classification, the traditional type of model for the 2-way crossed classification with covariate with s filled cells is

$$E(y_{ijk}) = \mu + \alpha_i + \beta_j + \gamma_{ij} + bx_{ijk} \,.$$

Numerous variants of this, such as

$$E(y_{ijk}) = \mu + \alpha_i + \beta_j + \gamma_{ij} + b_i x_{ijk} \,, \tag{9}$$

are discussed at length in Searle [1979]. Some of the computations for (9) applied to Exercise 12, Chapter 8 of Searle [1971, 2'nd and subsequent printings] are shown in Searle and Henderson [1978a, page 67], the Annotated Computer Output for SAS GLM. R( ) expressions for the sums of squares are shown in Table 9. For purposes of describing them, the model is best rewritten as

$$E(y_{ijk}) = \mu + \alpha_i + \beta_j + \gamma_{ij} + (b + b_i^*)x_{ij}$$

with

$$\underset{\sim}{b}^* = [b_1^* \;\; b_2^* \;\; \cdots \;\; b_a^*]' \,,$$

similar to that of the 1-way classification.

Table 9. Sums of Squares from SAS GLM for Fitting the Sequence X, X * A, A, B, A * B

Source	d.f.	Sums of Squares			
		TYPE I	TYPE II	TYPE III	TYPE IV
X	1	$R(b\|\mu)$	$R(b\|\mu,\alpha,\beta,\gamma)$	$R^*(\dot{b}\|\dot{\mu},\dot{\alpha},\dot{\beta},\dot{\gamma},\underset{\sim}{\dot{b}}^*)_\Sigma$	Same as TYPE III
X * A	a-1	$R(\underset{\sim}{b}^*\|\mu,b)$	$R(\underset{\sim}{b}^*\|\mu,\alpha,\beta,\gamma,b)$	$R^*(\underset{\sim}{\dot{b}}^*\|\dot{\mu},\dot{\alpha},\dot{\beta},\dot{\gamma},\dot{b})_\Sigma$	Same as TYPE III
A	a-1	$R(\alpha\|\mu,b,\underset{\sim}{b}^*)$	$R(\alpha\|\mu,\beta,b,\underset{\sim}{b}^*)$	$R^*(\dot{\alpha}\|\dot{\mu},\dot{\beta},\dot{\gamma},\dot{b},\underset{\sim}{\dot{b}}^*)_\Sigma$	Sums of squares for testing α-based and β-based contrasts, as defined in Searle [1979]; e.g., an α-based contrast might utilize $\bar{y}_{11\cdot} - \bar{y}_{21\cdot}$ and $\bar{y}_{13\cdot} + \bar{y}_{14\cdot} - \bar{y}_{33\cdot} - \bar{y}_{34\cdot}$.
B	b-1	$R(\beta\|\mu,\alpha,b,\underset{\sim}{b}^*)$	$R(\beta\|\mu,\alpha,b,\underset{\sim}{b}^*)$	$R^*(\dot{\beta}\|\dot{\mu},\dot{\alpha},\dot{\gamma},\dot{b},\underset{\sim}{\dot{b}}^*)_\Sigma$	
A * B	s-a-b-1	$R(\gamma\|\mu,\alpha,\beta,b,\underset{\sim}{b}^*)$	Same as TYPE I	Same as TYPE I	Same as TYPE I

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